

## REFLEXIVE POLYTOPES ARISING FROM PERFECT GRAPHS

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ABSTRACT. Reflexive polytopes form one of the distinguished classes of lattice polytopes. Especially reflexive polytopes which possess the integer decomposition property are of interest. In the present paper, by virtue of the algebraic technique on Gröbner bases, a new class of reflexive polytopes arising from perfect graphs which possess the integer decomposition property will be presented. Furthermore, the Ehrhart  $\delta$ -polynomials of these reflexive polytopes will be studied.

## BACKGROUND

The reflexive polytope is one of the keywords belonging to the current trends on the research of convex polytopes. In fact, many authors have been studied reflexive polytopes from viewpoints of combinatorics, commutative algebra and algebraic geometry. We are especially interested in reflexive polytopes which possess the integer decomposition property. It is known that in each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence ([11]) and all of them are known up to dimension 4 ([10]). Moreover, every lattice polytope is a face of a reflexive polytope ([3]).

To find new classes of reflexive polytopes is one of the most important problem. For example, the following classes of reflexive polytopes which possess the integer decomposition property are known:

- Centrally symmetric configurations ([14]).
- Reflexive polytopes arising from the order polytopes and the chain polytopes of finite partially ordered sets ([8, 9]).
- Reflexive polytopes arising from the stable sets polytopes of perfect graphs ([15]).

Following the previous work [15] the present paper discusses a new class of reflexive polytopes arising from perfect graphs.

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## 1. PERFECT GRAPHS AND REFLEXIVE POLYTOPES

Recall that a *lattice polytope* is a convex polytope all of whose vertices have integer coordinates. We say that a lattice polytope  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  is *reflexive* if the origin of  $\mathbb{R}^d$  belongs to the interior of  $\mathcal{P}$  and if the dual polytope

$$\mathcal{P}^\vee = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1, \forall \mathbf{y} \in \mathcal{P}\}$$

is again a lattice polytope. Here  $\langle \mathbf{x}, \mathbf{y} \rangle$  is the canonical inner product of  $\mathbb{R}^d$ .

Let  $G$  be a finite simple graph on the vertex set  $[d] = \{1, \dots, d\}$  and  $E(G)$  the set of edges of  $G$ . (A finite graph  $G$  is called simple if  $G$  possesses no loop and no multiple edge.) A subset  $W \subset [d]$  is called *stable* if, for all  $i$  and  $j$  belonging to  $W$  with  $i \neq j$ , one has  $\{i, j\} \notin E(G)$ . A *clique* of  $G$  is a subset  $W \subset [d]$  which is a stable set of the complementary graph  $\overline{G}$  of  $G$ . The *chromatic number* of  $G$  is the smallest integer  $t \geq 1$  for which there exist stable set  $W_1, \dots, W_t$  of  $G$  with  $[d] = W_1 \cup \dots \cup W_t$ . A finite simple graph  $G$  is said to be *perfect* ([2]) if, for any induced subgraph  $H$  of  $G$  including  $G$  itself, the chromatic number of  $H$  is equal to the maximal cardinality of cliques of  $H$ . The complementary graph of a perfect graph is perfect ([2]).

Let  $\mathbf{e}_1, \dots, \mathbf{e}_d$  denote the standard coordinate unit vectors of  $\mathbb{R}^d$ . Given a subset  $W \subset [d]$ , we may associate  $\rho(W) = \sum_{j \in W} \mathbf{e}_j \in \mathbb{R}^d$ . In particular  $\rho(\emptyset)$  is the origin  $\mathbf{0}_d$  of  $\mathbb{R}^d$ . Let  $S(G)$  denote the set of stable sets of  $G$ . One has  $\emptyset \in S(G)$  and  $\{i\} \in S(G)$  for each  $i \in [d]$ . The *stable set polytope*  $\mathcal{Q}_G \subset \mathbb{R}^d$  of  $G$  is the  $(0, 1)$ -polytope which is the convex hull of  $\{\rho(W) : W \in S(G)\}$  in  $\mathbb{R}^d$ . One has  $\dim \mathcal{Q}_G = d$ .

Let  $\mathcal{P} \subset \mathbb{R}^d$  be an arbitrary lattice polytope. We say that  $\mathcal{P} \subset \mathbb{R}^d$  possesses the *integer decomposition property* if, for each integer  $n \geq 1$  and for each  $\mathbf{a} \in n\mathcal{P} \cap \mathbb{Z}^d$ , where  $n\mathcal{P} = \{n\mathbf{a} : \mathbf{a} \in \mathcal{P}\}$ , there exist  $\mathbf{a}_1, \dots, \mathbf{a}_n$  belonging to  $\mathcal{P} \cap \mathbb{Z}^d$  with  $\mathbf{a} = \mathbf{a}_1 + \dots + \mathbf{a}_n$ .

Given lattice polytopes  $\mathcal{P} \subset \mathbb{R}^d$  and  $\mathcal{Q} \subset \mathbb{R}^d$  of dimension  $d$ , we introduce the lattice polytopes  $\Gamma(\mathcal{P}, \mathcal{Q}) \subset \mathbb{R}^d$  and  $\Omega(\mathcal{P}, \mathcal{Q}) \subset \mathbb{R}^{d+1}$  with

$$\Gamma(\mathcal{P}, \mathcal{Q}) = \text{conv}\{\mathcal{P} \cup (-\mathcal{Q})\},$$

$$\Omega(\mathcal{P}, \mathcal{Q}) = \text{conv}\{(\mathcal{P} \times \{1\}) \cup (-\mathcal{Q} \times \{-1\})\}.$$

Here  $\mathcal{P} \times \{1\} = \{(\mathbf{a}, 1) \in \mathbb{R}^{d+1} : \mathbf{a} \in \mathcal{P}\}$ .

It is natural to ask when  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  is reflexive and when  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  is reflexive, where  $G_1$  and  $G_2$  are finite simple graph on  $[d]$ . The former is completely solved by [15] and the later is studied in this paper. In fact,

**Theorem 1.1.** *Let  $G_1$  and  $G_2$  be finite simple graph on  $[d]$ .*

- (a) ([15]) *The following conditions are equivalent:*
  - (i) *The lattice polytope  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  is reflexive;*
  - (ii) *The lattice polytope  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  is reflexive and possesses the integer decomposition property;*

- (iii) Both  $G_1$  and  $G_2$  are perfect.
- (b) The following conditions are equivalent:
  - (i) The lattice polytope  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  possesses the integer decomposition property;
  - (ii) The lattice polytope  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  is reflexive and possesses the integer decomposition property;
  - (iii) Both  $G_1$  and  $G_2$  are perfect.

A proof of part (b) will be achieved in Section 2. It would, of course, be of interest to find a complete characterization for  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  to be reflexive.

If  $G_1$  and  $G_2$  are not perfect,  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  may not possess the integer decomposition property (Example 4.1 and 4.2). Furthermore, if  $G_1$  and  $G_2$  are not perfect,  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  may not be reflexive (Example 4.2 and 4.3).

We now turn to the discussion of Ehrhart  $\delta$ -polynomials of  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  and  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ . Let, in general,  $\mathcal{P} \subset \mathbb{R}^d$  be a lattice polytope of dimension  $d$ . The *Ehrhart  $\delta$ -polynomial* of  $\mathcal{P}$  is the polynomial

$$\delta(\mathcal{P}, \lambda) = (1 - \lambda)^{d+1} \left[ 1 + \sum_{n=1}^{\infty} |n\mathcal{P} \cap \mathbb{Z}^d| \lambda^n \right]$$

in  $\lambda$ . Each coefficient of  $\delta(\mathcal{P}, \lambda)$  is nonnegative integer and the degree of  $\delta(\mathcal{P}, \lambda)$  is at most  $d$ . In addition  $\delta(\mathcal{P}, 1)$  coincides with the normalized volume of  $\mathcal{P}$ , denoted by  $\text{vol}(\mathcal{P})$ . Refer the reader to [5, Part II] for the detailed information about Ehrhart  $\delta$ -polynomials.

The *suspension* of a finite simple graph  $G$  on  $[d]$  is the finite simple graph  $\widehat{G}$  on  $[d+1]$  with  $E(\widehat{G}) = E(G) \cup \{\{i, d+1\} : i \in [d]\}$ .

**Theorem 1.2.** *Let  $G_1$  and  $G_2$  be finite perfect simple graph on  $[d]$ . Then one has*

$$\delta(\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2}), \lambda) = \delta(\Gamma(\mathcal{Q}_{\widehat{G_1}}, \mathcal{Q}_{\widehat{G_2}}), \lambda) = (1 + \lambda)\delta(\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2}), \lambda).$$

*Thus in particular*

$$\text{vol}(\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})) = \text{vol}(\Gamma(\mathcal{Q}_{\widehat{G_1}}, \mathcal{Q}_{\widehat{G_2}})) = 2 \cdot \text{vol}(\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})).$$

A proof of Theorem 1.2 will be given in Section 3. Even though the Ehrhart  $\delta$ -polynomial of  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  coincides with that of  $\Gamma(\mathcal{Q}_{\widehat{G_1}}, \mathcal{Q}_{\widehat{G_2}})$ ,  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  may not be unimodularly equivalent to  $\Gamma(\mathcal{Q}_{\widehat{G_1}}, \mathcal{Q}_{\widehat{G_2}})$  (Example 4.4).

## 2. SQUAREFREE GRÖBNER BASIS

In this section, we prove Theorem 1.1 by using the theory of Gröbner bases and toric ideals. At first, we recall basic materials and notation on toric ideals. Let  $K[\mathbf{t}^{\pm 1}, s] = K[t_1^{\pm 1}, \dots, t_d^{\pm 1}, s]$  denote the Laurent polynomial ring in  $d+1$  variables over a field  $K$ . For an integer vector  $\mathbf{a} = [a_1, \dots, a_d]^T \in \mathbb{Z}^d$ , the transpose of  $[a_1, \dots, a_d]$ ,  $\mathbf{t}^{\mathbf{a}} s$  is the Laurent monomial  $t_1^{a_1} \cdots t_d^{a_d} s \in K[\mathbf{t}^{\pm 1}, s]$ . Given an integer

$d \times n$  matrix  $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ , where  $\mathbf{a}_j = [a_{1j}, \dots, a_{dj}]^\top$  is the  $j$ th column of  $A$ , then we define the *toric ring*  $K[A]$  of  $A$  as follows:

$$K[A] = K[\mathbf{t}^{\mathbf{a}_1}s, \dots, \mathbf{t}^{\mathbf{a}_n}s] \subset K[\mathbf{t}^{\pm 1}, s].$$

Let  $K[\mathbf{x}] = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $K$  and define the surjective ring homomorphism  $\pi : K[\mathbf{x}] \rightarrow K[A]$  by setting  $\pi(x_j) = \mathbf{t}^{\mathbf{a}_j}s$  for  $j = 1, \dots, n$ . The *toric ideal* of  $A$  is the kernel  $I_A$  of  $\pi$ . Let  $<$  be a monomial order on  $K[\mathbf{x}]$  and  $\text{in}_<(I_A)$  the initial ideal of  $I_A$  with respect to  $<$ . The initial ideal  $\text{in}_<(I_A)$  is called *squarefree* if  $\text{in}_<(I_A)$  is generated by squarefree monomials. Please refer [6, Chapters 1 and 5] for more details on Gröbner bases and toric ideals.

Let  $\mathbb{Z}_{\geq 0}^d$  denote the set of integer column vectors  $[a_1, \dots, a_d]^\top$  with each  $a_i \geq 0$ , and let  $\mathbb{Z}_{\geq 0}^{d \times n}$  denote the set of  $d \times n$  integer matrices  $(a_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}}$  with each  $a_{ij} \geq 0$ .

In [15], the concept that  $A \in \mathbb{Z}_{\geq 0}^{d \times n}$  and  $B \in \mathbb{Z}_{\geq 0}^{d \times m}$  are of *of harmony* is introduced. For an integer vector  $\mathbf{a} = [a_1, \dots, a_d]^\top \in \mathbb{Z}^d$ , let  $\mathbf{a}^{(+)} = [a_1^{(+)}, \dots, a_d^{(+)}]^\top$ ,  $\mathbf{a}^{(-)} = [a_1^{(-)}, \dots, a_d^{(-)}]^\top \in \mathbb{Z}_{\geq 0}^d$  where  $a_i^{(+)} = \max\{0, a_i\}$  and  $a_i^{(-)} = \max\{0, -a_i\}$ . Note that  $\mathbf{a} = \mathbf{a}^{(+)} - \mathbf{a}^{(-)}$  holds in general. Given  $A \in \mathbb{Z}_{\geq 0}^{d \times n}$  and  $B \in \mathbb{Z}_{\geq 0}^{d \times m}$  such that the zero vector  $\mathbf{0}_d = [0, \dots, 0]^\top \in \mathbb{Z}^d$  is a column of  $A$  and  $B$ , we say that  $A$  and  $B$  are of *harmony* if the following condition is satisfied: Let  $\mathbf{a}$  be a column of  $A$  and  $\mathbf{b}$  that of  $B$ . Let  $\mathbf{c} = \mathbf{a} - \mathbf{b} \in \mathbb{Z}^d$ . If  $\mathbf{c} = \mathbf{c}^{(+)} - \mathbf{c}^{(-)}$ , then  $\mathbf{c}^{(+)}$  is a column vector of  $A$  and  $\mathbf{c}^{(-)}$  is a column vector of  $B$ .

Now we prove the following theorem.

**Theorem 2.1.** *Let  $A = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in \mathbb{Z}_{\geq 0}^{d \times n}$  and  $B = [\mathbf{b}_1, \dots, \mathbf{b}_m] \in \mathbb{Z}_{\geq 0}^{d \times m}$ , where  $\mathbf{a}_n = \mathbf{b}_m = \mathbf{0}_d \in \mathbb{Z}^d$ , be of harmony. Let  $K[\mathbf{x}] = K[x_1, \dots, x_n]$  and  $K[\mathbf{y}] = K[y_1, \dots, y_m]$  be the polynomial rings over a field  $K$ . Suppose that  $\text{in}_{<_A}(I_A) \subset K[\mathbf{x}]$  and  $\text{in}_{<_B}(I_B) \subset K[\mathbf{y}]$  are squarefree with respect to reverse lexicographic orders  $<_A$  on  $K[\mathbf{x}]$  and  $<_B$  on  $K[\mathbf{y}]$  respectively satisfying the condition that*

- $x_i <_A x_j$  if  $\pi(x_i)$  divides  $\pi(x_j)$ ;
- $x_n$  is the smallest variable with respect to  $<_A$ .
- $y_m$  is the smallest variable with respect to  $<_B$ .

Let  $[-B, A]^*$  denote the  $(d+1) \times (n+m+1)$  integer matrix

$$\begin{bmatrix} -\mathbf{b}_1 & \cdots & -\mathbf{b}_m & \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{0}_d \\ -1 & \cdots & -1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

Then the toric ideal  $I_{[-B, A]^*}$  of  $[-B, A]^*$  possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column  $\mathbf{0}_{d+1} \in \mathbb{Z}^{d+1}$  of  $[-B, A]^*$ .

*Proof.* Let  $I_{[-B, A]^*} \subset K[\mathbf{x}, \mathbf{y}, z] = K[x_1, \dots, x_n, y_1, \dots, y_m, z]$  be the toric ideal of  $[-B, A]^*$  defined by the kernel of

$$\pi^* : K[\mathbf{x}, \mathbf{y}, z] \rightarrow K[[-B, A]^*] \subset K[t_1^{\pm 1}, \dots, t_{d+1}^{\pm 1}, s]$$

with  $\pi^*(z) = s$ ,  $\pi^*(x_i) = \mathbf{t}^{\mathbf{a}_i} t_{d+1} s$  for  $i = 1, \dots, n$  and  $\pi^*(y_j) = \mathbf{t}^{-\mathbf{b}_j} t_{d+1}^{-1} s$  for  $j = 1, \dots, m$ . Assume that the reverse lexicographic orders  $<_A$  and  $<_B$  are induced by the orderings  $x_n <_A \dots <_A x_1$  and  $y_m <_B \dots <_B y_1$ . Let  $<_{\text{rev}}$  be the reverse lexicographic order on  $K[\mathbf{x}, \mathbf{y}, z]$  induced by the ordering

$$z <_{\text{rev}} x_n <_{\text{rev}} \dots <_{\text{rev}} x_1 <_{\text{rev}} y_m <_{\text{rev}} \dots <_{\text{rev}} y_1.$$

In general, for an integer vector  $\mathbf{a} = [a_1, \dots, a_d]^\top \in \mathbb{Z}^d$ , we let  $\text{supp}(\mathbf{a}) = \{i : 1 \leq i \leq d, a_i \neq 0\}$ . Set the following:

$$\mathcal{E} = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m, \text{supp}(\mathbf{a}_i) \cap \text{supp}(\mathbf{b}_j) \neq \emptyset\}.$$

If  $\mathbf{c} = \mathbf{a}_i - \mathbf{b}_j$  with  $(i, j) \in \mathcal{E}$ , then it follows that  $\mathbf{c}^{(+)} \neq \mathbf{a}_i$  and  $\mathbf{c}^{(-)} \neq \mathbf{b}_j$ . Since  $A$  and  $B$  are of harmony, we know that  $\mathbf{c}^{(+)}$  is a column of  $A$  and  $\mathbf{c}^{(-)}$  is a column of  $B$ . It follows that  $f = x_i y_j - u$  ( $\neq 0$ ) belongs to  $I_{[-B, A]^*}$ , where

$$u = \begin{cases} x_k y_\ell & \text{if } \mathbf{c}^{(+)} = \mathbf{a}_k \text{ and } \mathbf{c}^{(-)} = \mathbf{b}_\ell, \\ z^2 & \text{if } \mathbf{c}^{(+)} = \mathbf{c}^{(-)} = \mathbf{0}_d. \end{cases}$$

If  $u = z^2$ , then  $\text{in}_{<_{\text{rev}}}(f) = x_i y_j$ , where  $\text{in}_{<_{\text{rev}}}(f)$  is the initial monomial of  $f \in K[\mathbf{x}, \mathbf{y}, z]$ . If  $u = x_k y_\ell$ , then since  $\pi(x_k)$  divides  $\pi(x_i)$ , one has  $x_k <_A x_i$  and  $\text{in}_{<_{\text{rev}}}(f) = x_i y_j$ . Hence

$$\{x_i y_j : (i, j) \in \mathcal{E}\} \subset \text{in}_{<_{\text{rev}}}(I_{[-B, A]^*}).$$

Moreover, it follows that  $x_n y_m - z^2 \in I_{[-B, A]^*}$  and  $x_n y_m \in \text{in}_{<_{\text{rev}}}(I_{[-B, A]^*})$ . We set

$$\mathcal{M} = \{x_n y_m\} \cup \{x_i y_j : (i, j) \in \mathcal{E}\} \cup \mathcal{M}_A \cup \mathcal{M}_B \quad (\subset \text{in}_{<_{\text{rev}}}(I_{[-B, A]^*})),$$

where  $\mathcal{M}_A$  (resp.  $\mathcal{M}_B$ ) is the minimal set of squarefree monomial generators of  $\text{in}_{<_A}(I_A)$  (resp.  $\text{in}_{<_B}(I_B)$ ). Let  $\mathcal{G}$  be a finite set of binomials belonging to  $I_{[-B, A]^*}$  with  $\mathcal{M} = \{\text{in}_{<_{\text{rev}}}(f) : f \in \mathcal{G}\}$ .

Now, we prove that  $\mathcal{G}$  is a Gröbner base of  $\text{in}_{<_{\text{rev}}}(I_{[-B, A]^*})$  with respect to  $<_{\text{rev}}$ . By the following fact ([13, (0.1), p. 1914]) on Gröbner bases, we must prove the following assertion: If  $u$  and  $v$  are monomials belonging to  $K[\mathbf{x}, \mathbf{y}, z]$  with  $u \neq v$  such that  $u \notin (\text{in}_{<}(g) : g \in \mathcal{G})$  and  $v \notin (\text{in}_{<}(g) : g \in \mathcal{G})$ , then  $\pi^*(u) \neq \pi^*(v)$ .

Suppose that there exists a nonzero irreducible binomial  $g = u - v$  belonging to  $I_{[-B, A]^*}$  such that  $u \notin (\text{in}_{<}(g) : g \in \mathcal{G})$  and  $v \notin (\text{in}_{<}(g) : g \in \mathcal{G})$ . Write

$$u = \left( \prod_{p \in P} x_p^{i_p} \right) \left( \prod_{q \in Q} y_q^{j_q} \right), \quad v = z^\alpha \left( \prod_{p' \in P'} x_{p'}^{i'_{p'}} \right) \left( \prod_{q' \in Q'} y_{q'}^{j'_{q'}} \right),$$

where  $P$  and  $P'$  are subsets of  $[n]$ , where  $Q$  and  $Q'$  are subsets of  $[m]$ , where  $\alpha$  is a nonnegative integer, and where each of  $i_p, j_q, i'_{p'}, j'_{q'}$  is a positive integer. Since  $g = u - v$  is irreducible, one has  $P \cap P' = Q \cap Q' = \emptyset$ . Furthermore, by the fact

that each of  $x_i y_j$  with  $(i, j) \in \mathcal{E}$  can divide neither  $u$  nor  $v$ , it follows that

$$\left( \bigcup_{p \in P} \text{supp}(\mathbf{a}_p) \right) \cap \left( \bigcup_{q \in Q} \text{supp}(\mathbf{b}_q) \right) = \left( \bigcup_{p' \in P'} \text{supp}(\mathbf{a}_{p'}) \right) \cap \left( \bigcup_{q' \in Q'} \text{supp}(\mathbf{b}_{q'}) \right) = \emptyset.$$

Hence, since  $\pi^*(u) = \pi^*(v)$ , it follows that

$$\sum_{p \in P} i_p \mathbf{a}_p = \sum_{p' \in P'} i'_{p'} \mathbf{a}_{p'}, \quad \sum_{q \in Q} j_q \mathbf{b}_q = \sum_{q' \in Q'} j'_{q'} \mathbf{b}_{q'}.$$

Let  $\xi = \sum_{p \in P} i_p$ ,  $\xi' = \sum_{p' \in P'} i'_{p'}$ ,  $\nu = \sum_{q \in Q} j_q$ , and  $\nu' = \sum_{q' \in Q'} j'_{q'}$ . Then  $\xi + \nu = \xi' + \nu' + \alpha$ . Since  $\alpha \geq 0$ , it follows that either  $\xi \geq \xi'$  or  $\nu \geq \nu'$ . Assume that  $\xi > \xi'$ . Then

$$h = \prod_{p \in P} x_p^{i_p} - x_n^{\xi - \xi'} \left( \prod_{p' \in P'} x_{p'}^{i'_{p'}} \right)$$

belongs to  $I_A$  and  $I_{[-B, A]^*}$ . If  $h \neq 0$ , then  $\text{in}_{<_A}(h) = \text{in}_{<_{\text{rev}}}(h) = \prod_{p \in P} x_p^{i_p}$  divides  $u$ , a contradiction. Hence  $P = \{n\}$  and  $Q = \emptyset$ . If  $\xi = \xi'$ , then the binomial

$$h_0 = \prod_{p \in P} x_p^{i_p} - \prod_{p' \in P'} x_{p'}^{i'_{p'}}$$

belongs to  $I_A$  and  $I_{[-B, A]^*}$ . Moreover, if  $h_0 \neq 0$ , then either  $\prod_{p \in P} x_p^{i_p}$  or  $\prod_{p' \in P'} x_{p'}^{i'_{p'}}$  must belong to  $\text{in}_{<_A}(I_A)$  and  $\text{in}_{<_{\text{rev}}}(I_{[-B, A]^*})$ . This contradicts the fact that each of  $u$  and  $v$  can be divided by none of the monomials belonging to  $\mathcal{M}$ . Hence  $h_0 = 0$  and  $P = P' = \emptyset$ . Similarly,  $Q = \{m\}$  and  $Q' = \emptyset$ , or  $Q = Q' = \emptyset$ . Hence we know that  $g = x_n^k y_m^\ell - z^\alpha$ , where  $k$  and  $\ell$  are nonnegative integers. Since  $u$  cannot be divided  $x_n y_m$ , it follows that  $g = 0$ , contradiction. Therefore,  $\mathcal{G}$  is a Gröbner base of  $\text{in}_{<_{\text{rev}}}(I_{[-B, A]^*})$  with respect to  $<_{\text{rev}}$ .  $\square$

Now, we recall the following lemma.

**Lemma 2.2** ([7, Lemma 1.1]). *Let  $\mathcal{P} \subset \mathbb{R}^d$  be an lattice polytope such that the origin of  $\mathbb{R}^d$  is contained in its interior and  $\mathcal{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ . Suppose that any integer point in  $\mathbb{Z}^{d+1}$  is a linear integer combination of the integer points in  $\mathcal{P} \times \{1\}$  and there exists an ordering of the variables  $x_{i_1} < \dots < x_{i_n}$  for which  $\mathbf{a}_{i_1} = \mathbf{0}_d$  such that the initial ideal  $\text{in}_{<}(I_A)$  of the toric ideal  $I_A$  with respect to the reverse lexicographic order  $<$  on the polynomial ring  $K[x_1, \dots, x_n]$  induced by the ordering is squarefree, where  $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ . Then  $\mathcal{P}$  is a reflexive polytope which possesses the integer decomposition property.*

By Theorem 2.1 and this lemma, we obtain the following corollary.

**Corollary 2.3.** *Work with the same situation as in Theorem 2.1. Let  $\mathcal{P} \subset \mathbb{R}^{d+1}$  be the lattice polytope which is the convex hull of*

$$\{-(\mathbf{b}_1, 1), \dots, -(\mathbf{b}_m, 1), (\mathbf{a}_1, 1), \dots, (\mathbf{a}_n, 1)\}.$$

Suppose that  $\mathbf{0}_{d+1} \in \mathbb{Z}^{d+1}$  belongs to the interior of  $\mathcal{P}$  and any integer point in  $\mathbb{Z}^{d+2}$  is a linear integer combination of the integer points in  $\mathcal{P} \times \{1\}$ . Then  $\mathcal{P}$  is a reflexive polytope which possesses the integer decomposition property.

Recall that an integer matrix  $A$  is *compressed* ([12], [16]) if the initial ideal of the toric ideal  $I_A$  is squarefree with respect to any reverse lexicographic order.

Finally, we prove Theorem 1.1.

*Proof of Theorem 1.1.* For any perfect graph  $G$ ,  $A_{S(G)}$  is compressed ([12, Example 1.3 (c)]). Let  $\mathcal{P} \subset \mathbb{R}^{d+1}$  be the convex hull of  $\{\pm(\mathbf{e}_1 + \mathbf{e}_{d+1}), \dots, \pm(\mathbf{e}_d + \mathbf{e}_{d+1}), \pm\mathbf{e}_{d+1}\}$ . It is easy to show that  $\mathbf{0}_{d+1} \in \mathbb{Z}^{d+1}$  belongs to the interior of  $\mathcal{P}$  and any integer point in  $\mathbb{Z}^{d+2}$  is a linear integer combination of the integer points in  $\mathcal{P} \times \{1\}$ . Moreover, we have  $\mathcal{P} \subset \Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ . Hence, by Corollary 2.3,  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  is a reflexive polytope which possesses the integer decomposition property.

Next, we prove that if  $G_1$  is not perfect, then  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  does not possess the integer decomposition property. Assume that  $G_1$  is not perfect and  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  possesses the integer decomposition property. By the strong perfect graph theorem ([2]),  $G_1$  possesses either an odd hole or an odd antihole, where an odd hole is an induced odd cycle of length  $\geq 5$  and an odd antihole is the complementary graph of an odd hole. Suppose that  $G_1$  possesses an odd hole  $C$  of length  $2\ell + 1$ , where  $\ell \geq 2$ . We may assume that the vertex set of  $C$  is  $[2\ell + 1]$  and the edge set is  $\{\{i, i+1\} : 1 \leq i \leq 2\ell\} \cup \{1, 2\ell + 1\}$ . Then the maximal stable sets of  $C$  is

$$S_1 = \{1, 3, \dots, 2\ell - 1\}, S_2 = \{2, 4, \dots, 2\ell\}, \dots, S_{2\ell+1} = \{2\ell + 1, 2, 4, \dots, 2\ell - 2\}$$

and each  $i \in [2\ell + 1]$  appears  $\ell$  times in the above list. For  $1 \leq i \leq 2\ell + 1$ , we set  $\mathbf{v}_i = \sum_{j \in S_i} \mathbf{e}_j + \mathbf{e}_{d+1}$ . Then one has

$$\mathbf{a} = \frac{\mathbf{v}_1 + \dots + \mathbf{v}_{2\ell+1} + (-\mathbf{e}_{d+1})}{\ell} = \mathbf{e}_1 + \dots + \mathbf{e}_{2\ell+1} + 2\mathbf{e}_{d+1}.$$

Since  $2 < (2\ell + 2)/\ell \leq 3$ ,  $\mathbf{a} \in 3\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ . Hence there exist  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2}) \cap \mathbb{Z}^{d+1}$  such that  $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$ . Then we may assume that  $\mathbf{a}_1, \mathbf{a}_2 \in \mathcal{Q}_C \times \{1\}$  and  $\mathbf{a}_3 = \mathbf{0}_{d+1}$ . However, since the maximal cardinality of stable sets of  $C$  equals  $\ell$ , this is in contradiction.

Suppose that  $G_1$  possesses an odd antihole  $C$  of length  $2\ell + 1$ , where  $\ell \geq 2$ . Similarly, we may assume that the vertex set of  $C$  is  $[2\ell + 1]$  and the edge set is  $\{\{i, i+1\} : 1 \leq i \leq 2\ell\} \cup \{1, 2\ell + 1\}$ . Then the maximal stable sets of  $\overline{C}$  is the edges of  $C$ . For  $1 \leq i \leq 2\ell$ , we set  $\mathbf{w}_i = \mathbf{e}_i + \mathbf{e}_{i+1} + \mathbf{e}_{d+1}$  and set  $\mathbf{w}_{2\ell+1} = \mathbf{e}_1 + \mathbf{e}_{2\ell+1} + \mathbf{e}_{d+1}$ . Then one has

$$\mathbf{b} = \frac{\mathbf{w}_1 + \dots + \mathbf{w}_{2\ell+1} + (-\mathbf{e}_{d+1})}{2} = \mathbf{e}_1 + \dots + \mathbf{e}_{2\ell+1} + \ell\mathbf{e}_{d+1}$$

and  $\mathbf{b} \in (\ell + 1)\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ . Hence there exist  $\mathbf{b}_1, \dots, \mathbf{b}_{\ell+1} \in \Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2}) \cap \mathbb{Z}^{d+1}$  such that  $\mathbf{b} = \mathbf{b}_1 + \dots + \mathbf{b}_{\ell+1}$ . Then we may assume that  $\mathbf{b}_1, \dots, \mathbf{b}_\ell \in \mathcal{Q}_{\overline{C}} \times \{1\}$



and  $\mathbf{b}_{\ell+1} = \mathbf{0}_{d+1}$ . However, since the maximal cardinality of stable sets of  $\overline{\mathcal{C}}$  equals 2, this is in contradiction.

Therefore, if  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  possesses the integer decomposition property, then  $G_1$  and  $G_2$  are perfect, as desired.  $\square$

### 3. EHRHART $\delta$ -POLYNOMIALS

In this section, we consider combinatorial properties of these polytopes, especially, the Ehrhart  $\delta$ -polynomials and the volume of  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  and  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ . Let  $\mathcal{P} \subset \mathbb{R}^d$  be an integral convex polytope of dimension  $d$  with  $\mathcal{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ . Set  $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ . We define the toric ring  $K[\mathcal{P}]$  and the toric ideal  $I_{\mathcal{P}}$  of  $\mathcal{P}$  by  $K[A]$  and  $I_A$ . In order to prove Theorem 1.2, we use the following facts.

- If  $\mathcal{P}$  possesses the integer decomposition property, then the *Ehrhart polynomial*  $|n\mathcal{P} \cap \mathbb{Z}^d|$  of  $\mathcal{P}$  is equal to the Hilbert polynomial of the toric ring  $K[\mathcal{P}]$ .
- Let  $S$  be a polynomial ring and  $I \subset S$  be a graded ideal of  $S$ . Let  $<$  be a monomial order on  $S$ . Then  $S/I$  and  $S/\text{in}_{<}(I)$  have the same Hilbert function. (see [4, Corollary 6.1.5])

Now, we prove the following Theorem.

**Theorem 3.1.** *Work with the same situation as in Theorem 2.1. Let  $\mathcal{P} \subset \mathbb{R}^d$  be the lattice polytope which is the convex hull of  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  and  $\mathcal{Q} \subset \mathbb{R}^d$  the lattice polytope which is the convex hull of  $\{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ . Then we obtain*

$$\delta(\Omega(\mathcal{P}, \mathcal{Q}), \lambda) = (1 + \lambda)\delta(\Gamma(\mathcal{P}, \mathcal{Q}), \lambda).$$

In particular,

$$\text{vol}(\Omega(\mathcal{P}, \mathcal{Q})) = 2 \cdot \text{vol}(\Gamma(\mathcal{P}, \mathcal{Q})).$$

*Proof.* Set  $\mathcal{R} = \text{conv}(\Gamma(\mathcal{P}, \mathcal{Q}) \times \{0\}, \pm \mathbf{e}_{d+1})$ . Then it follows from [1, Theorem 1.4] that  $\delta(\mathcal{R}, \lambda) = (1 + \lambda)\delta(\Gamma(\mathcal{P}, \mathcal{Q}), \lambda)$ . Moreover, by [15, Theorem 1.1] and Theorem 2.1,  $\mathcal{R}$  and  $\Omega(\mathcal{P}, \mathcal{Q})$  possess the integer decomposition property. Hence we should show that  $K[\mathcal{R}]$  and  $K[\Omega(\mathcal{P}, \mathcal{Q})]$  have the same Hilbert function.

Now, use the same notation as in the proof of Theorem 2.1. Then we have

$$\frac{K[\mathbf{x}, \mathbf{y}, z]}{\text{in}_{<_{\text{ref}}}(\mathcal{I}_{\Omega(\mathcal{P}, \mathcal{Q})})} = \frac{K[\mathbf{x}, \mathbf{y}, z]}{(\mathcal{M})}.$$

Set

$$\mathbf{a}'_i = \begin{cases} (\mathbf{a}_i, 0), & 1 \leq i \leq n-1, \\ \mathbf{e}_{d+1}, & i = n, \\ \mathbf{0}_{d+1}, & i = n+1, \end{cases} \quad \text{and} \quad \mathbf{b}'_j = \begin{cases} (\mathbf{b}_j, 0), & 1 \leq j \leq m-1, \\ \mathbf{e}_{d+1}, & j = m, \\ \mathbf{0}_{d+1}, & j = m+1. \end{cases}$$

Then it is easy to show that  $A' = [\mathbf{a}'_1, \dots, \mathbf{a}'_{n+1}]$  and  $B' = [\mathbf{b}'_1, \dots, \mathbf{b}'_{m+1}]$  are of harmony. Moreover,  $\text{in}_{<_{B'}}(I_{B'}) \subset K[y_1, \dots, y_{m+1}]$  and  $\text{in}_{<_{A'}}(I_{A'}) \subset K[x_1, \dots, x_{n+1}]$



are squarefree with respect to reverse lexicographic orders  $<_{A'}$  on  $K[x_1, \dots, x_{n+1}]$  and  $<_{B'}$  on  $K[y_1, \dots, y_{m+1}]$  induced by the orderings  $x_{n+1} <_{A'} x_n <_{A'} \dots <_{A'} x_1$  and  $y_{m+1} <_{B'} y_m <_{B'} \dots <_{B'} y_1$ . Now, we introduce the following

$$\mathcal{E}' = \{ (i, j) : 1 \leq i \leq n, 1 \leq j \leq m, \text{supp}(\mathbf{a}'_i) \cap \text{supp}(\mathbf{b}'_j) \neq \emptyset \}.$$

Then we have  $\mathcal{E}' = \mathcal{E} \cup \{(n, m)\}$ . Let  $\mathcal{M}_{A'}$  (resp.  $\mathcal{M}_{B'}$ ) be the minimal set of squarefree monomial generators of  $\text{in}_{<_{A'}}(I_{A'})$  (resp.  $\text{in}_{<_{B'}}(I_{B'})$ ). Then it follows that  $\mathcal{M}_{A'} = \mathcal{M}_A$  and  $\mathcal{M}_{B'} = \mathcal{M}_B$ . This says that  $\mathcal{M} = \mathcal{E}' \cup \mathcal{M}_{A'} \cup \mathcal{M}_{B'}$ . By the proof of [15, Theorem 1.1], we obtain  $\text{in}_{<_{\text{rev}}}(I_{\mathcal{R}}) = (\mathcal{M}) \subset K[\mathbf{x}, \mathbf{y}, z]$ . Hence it follows that

$$\frac{K[\mathbf{x}, \mathbf{y}, z]}{\text{in}_{<_{\text{rev}}}(I_{\Omega(\mathcal{P}, \mathcal{Q})})} = \frac{K[\mathbf{x}, \mathbf{y}, z]}{\text{in}_{<_{\text{rev}}}(I_{\mathcal{R}})}.$$

Therefore,  $K[\mathcal{R}]$  and  $K[\Omega(\mathcal{P}, \mathcal{Q})]$  have the same Hilbert function, as desired.  $\square$

Now, we prove Theorem 1.2.

*Proof of Theorem 1.2.* For any finite simple graph  $G$  on  $[d]$ , we have  $S(\widehat{G}) = S(G) \cup \{d+1\}$ . Hence it follows that  $\Omega(\mathcal{Q}_{\widehat{G}_1}, \mathcal{Q}_{\widehat{G}_2}) = \text{conv}(\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_1}) \times \{0\}, \pm \mathbf{e}_{d+1})$ . Therefore, by Theorem 3.1, we obtain

$$\delta(\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2}), \lambda) = \delta(\Gamma(\mathcal{Q}_{\widehat{G}_1}, \mathcal{Q}_{\widehat{G}_2}), \lambda) = (1 + \lambda)\delta(\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2}), \lambda),$$

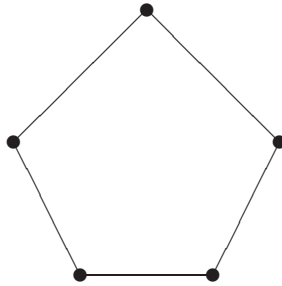
as desired.  $\square$

#### 4. EXAMPLES

In this section, we give some curious examples of  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  and  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$ . At first, the following example says that even though  $G_1$  and  $G_2$  are not perfect,  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  may be reflexive.

**Example 4.1.** Let  $G$  be the finite simple graph as follows:

$G$ :



Namely,  $G$  is a cycle of length 5. Then  $G$  is not perfect. Hence  $\Gamma(\mathcal{Q}_G, \mathcal{Q}_G)$  is not reflexive. However,  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  is reflexive. In fact, we have

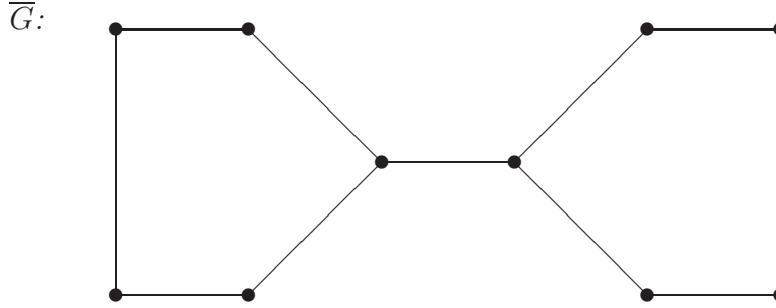
$$\delta(\Gamma(\mathcal{Q}_G, \mathcal{Q}_G), \lambda) = 1 + 15\lambda + 60\lambda^2 + 62\lambda^3 + 15\lambda^4 + \lambda^5,$$

$$\delta(\Omega(\mathcal{Q}_G, \mathcal{Q}_G), \lambda) = 1 + 16\lambda + 75\lambda^2 + 124\lambda^3 + 75\lambda^4 + 16\lambda^5 + \lambda^6.$$

Moreover,  $\Gamma(\mathcal{Q}_G, \mathcal{Q}_G)$  possesses the integer decomposition property, but  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  does not possess the integer decomposition property.

For this example,  $\Gamma(\mathcal{Q}_G, \mathcal{Q}_G)$  possesses the integer decomposition property. Next example says that if  $G_1$  and  $G_2$  are not perfect,  $\Gamma(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  may not possess the integer decomposition property.

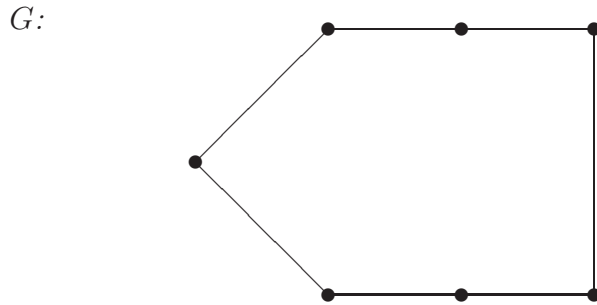
**Example 4.2.** Let  $G$  be a finite simple graph whose complementary graph  $\overline{G}$  is as follows:



Then  $G$  is not perfect. Hence  $\Gamma(\mathcal{Q}_G, \mathcal{Q}_G)$  is not reflexive. However,  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  is reflexive. Moreover, in this case,  $\Gamma(\mathcal{Q}_G, \mathcal{Q}_G)$  and  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  do not possess the integer decomposition property.

For any finite simple graph  $G$  with at most 6 vertices,  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  is always reflexive. However, in the case of finite simple graphs with more than 6 vertices, we obtain other result.

**Example 4.3.** Let  $G$  be the finite simple graph as follows:



Namely,  $G$  is a cycle of length 7. Then  $G$  is not perfect. Hence  $\Gamma(\mathcal{Q}_G, \mathcal{Q}_G)$  is not reflexive. Moreover,  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  is not reflexive. In fact, we have

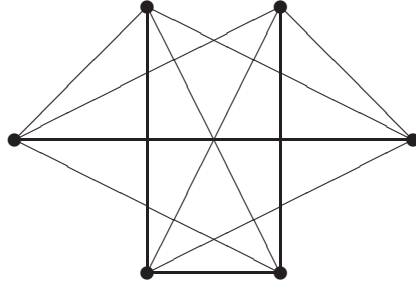
$$\delta(\Gamma(\mathcal{Q}_G, \mathcal{Q}_G), \lambda) = 1 + 49\lambda + 567\lambda^2 + 1801\lambda^3 + 1799\lambda^4 + 569\lambda^5 + 49\lambda^6 + \lambda^7,$$

$$\delta(\Omega(\mathcal{Q}_G, \mathcal{Q}_G), \lambda) = 1 + 50\lambda + 616\lambda^2 + 2370\lambda^3 + 3598\lambda^4 + 2368\lambda^5 + 618\lambda^6 + 50\lambda^7 + \lambda^8.$$

Finally, we show that even though the Ehrhart  $\delta$ -polynomial of  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  coincides with that of  $\Gamma(\mathcal{Q}_{\widehat{G}_1}, \mathcal{Q}_{\widehat{G}_2})$ ,  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  may not be unimodularly equivalent to  $\Gamma(\mathcal{Q}_{\widehat{G}_1}, \mathcal{Q}_{\widehat{G}_2})$ .

**Example 4.4.** Let  $G$  be the finite simple graph as follows:

$G$ :



Namely,  $G$  is a  $(2,2,2)$ -complete multipartite graph. Then  $G$  is perfect. Hence we know that  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  and  $\Gamma(\mathcal{Q}_{\widehat{G}}, \mathcal{Q}_{\widehat{G}})$  have the same Ehrhart  $\delta$ -polynomial and the same volume. However,  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  has 54 facets and  $\Gamma(\mathcal{Q}_{\widehat{G}}, \mathcal{Q}_{\widehat{G}})$  has 432 facets. Hence,  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$  and  $\Gamma(\mathcal{Q}_{\widehat{G}}, \mathcal{Q}_{\widehat{G}})$  are not unimodularly equivalent. Moreover, for any simple graph  $G'$  on  $\{1, \dots, 7\}$  except for  $\widehat{G}$ , the Ehrhart  $\delta$ -polynomial of  $\Gamma(\mathcal{Q}_{G'}, \mathcal{Q}_{G'})$  is not equal to that of  $\Omega(\mathcal{Q}_G, \mathcal{Q}_G)$ . This implies that the class of  $\Omega(\mathcal{Q}_{G_1}, \mathcal{Q}_{G_2})$  is a new class of reflexive polytopes.

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